Tien [6] has obtained good agreement with saturated nucleate pool boiling experiments using the basic heat-transfer relation

$$Nu = \text{constant } Pr^{\frac{1}{3}} Re^{\frac{1}{2}} \quad (1)$$

For gas injection the Reynolds number is given by $V_\infty L/\nu$ where V_∞ is the velocity of the injected gas, L is a characteristic length and ν is the kinematic viscosity. Thus equation (1) becomes

$$h = Ck\,Pr^{\frac{1}{3}}\left(\frac{V_\infty}{\nu}\right)^{\frac{1}{2}} \quad (2)$$

where C is a constant which has the dimensions of (length)$^{-\frac{1}{2}}$. In Fig. 1 the experimental data (1) for the heat-transfer coefficient as a function of the gas injection velocity is presented. The gas injection velocity was defined as the volumetric gas flow rate divided by the area of the heat-transfer surface. Good agreement with the data for all the liquids tested is given by equation (5) with a value of 11·2 ft$^{-\frac{1}{2}}$ for C.

When there are important interaction effects, equation (5) is no longer valid. In Fig. 1, the experimental points with horizontal tags are in this range and are presented to emphasize this effect. Reference may also be made to Gose *et al.* [1]. For the small gas injection rates, free convection effects must also be considered.

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REMARK ON THE LAMINAR BOUNDARY LAYER WITH PRESCRIBED ENERGY FLUX

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INTRODUCTION

In reference 1 Sparrow and Lin present an interesting analysis of boundary layers with prescribed heat transfer and then apply their analysis to flows with simultaneous convective and radiative heat transfer. One problem considered by them is that of a plate with arbitrarily prescribed laminar heat transfer; for this they employed an approximate solution given by Eckert and Drake [2] to form an integral equation and adjusted a constant so as to match the exact results obtainable for the case of uniform heat flux. It is the purpose of the present paper to give in principle an exact solution for this problem for the case of simplified transport properties and of unity Prandtl number; we include the words "in principle" because the solution is given in terms of eigenfunctions, only the first 10 of which have been provided by Fox and Libby [3]. However, additional functions can be readily obtain side if dered.

Rather than develop the solution which would correspond immediately to that of reference 1, we prefer to exploit techniques widely used in the aerospace literature and to demonstrate the solution in a somewhat more general form. First, we carry out the analysis in terms of two transformed variables, the so-called Levy–Lees variables, η and s which are related to the usual x, y Cartesian coordinates of the boundary-layer theory by

$$s = \int_0^x \rho_e \mu_e u_e r^{2j} \, dx' \tag{1}$$

$$\eta = \rho_e u_e r^j (2s)^{-1/2} \int_0^y (\rho/\rho_e) \, dy' \tag{2}$$

In equation (1) we leave the properties of the external flow within the integral since, as we shall discuss below, in some cases our analysis will apply at least approximately to flows with pressure gradient, i.e. to flows with varying external streams. The presence of the index j in equation (1) implies we are considering either two-dimensional flows, $j = 0$, or axisymmetric flows, $j = 1$.

We next assume that the gas, which may be either homogeneous or heterogeneous, has at each x-wise station a product of mass density and viscosity μ independent of y so that $C \equiv (\rho\mu/\rho_e\mu_e) \simeq 1$ and that the velocity field is described by the Blasius solution in terms of η. We note that with $C = 1$ the velocity within the boundary layer over either a two-dimensional, flat surface or a cone with supersonic external flow is exactly described by this solution; in addition for "cold-wall" flows, i.e. those for which $\rho_e/\rho_w \ll 1$ and the Mach number in the external stream is low, the Blasius solution provides an approximate velocity distribution. In this latter case the integral in equation (1) must be evaluated before an explicit relation between s and x is known.

Finally, we consider the energy equation in terms of the stagnation enthalpy ratio, $g \equiv h_s/h_{s,e}$; for low-speed flows with gases having fixed composition and constant coefficients of specific heat, this ratio is simply T/T_e, but the wider utility of g is clear. We are thus including, in general, high-speed, chemically reacting flows. We do, however, restrict our attention to the case of unity Prandtl number and unity Lewis number; for the case of low-speed flows with constant properties such as considered in reference 1 the treatment of nonunity Prandtl number requires only the computation of new eigenfunctions for each Prandtl number of interest.*

DEVELOPMENT OF THE SOLUTION

With the above preparation we are able to proceed; the conservation equation for g is

$$Lg = \frac{\partial^2 g}{\partial \eta^2} + f_0 \frac{\partial g}{\partial \eta} - 2s f_0' \frac{\partial g}{\partial s} = 0 \tag{3}$$

where $f_0 = f_0(\eta)$ is the Blasius function defined by

$$f_0''' + f_0 f_0'' = 0$$

$$f_0(0) = f_0'(0) = 0; \quad f_0'(\infty) = 1$$

and with the important surface value

$$f_0''(0) = f_{0,w}'' = 0{\cdot}469600.$$

We are concerned with the convective energy flux at the surface; in terms of g and with the assumed transport properties the energy flux q_w which includes, in general, both thermal and diffusive parts, is given by

$$q_w = \rho_e \mu_e u_e h_{s,e} r^j (2s)^{-1/2} \left(\frac{\partial g}{\partial \eta}\right)_w. \tag{4}$$

For notational simplicity let $\tilde{q} \equiv (\partial g/\partial \eta)_w$ and note that if $\tilde{q} = \tilde{q}(s)$ is specified, and if the distribution of external flow properties is known, then through equation (4) so is $q_w = q_w(s)$.

We take as the initial and boundary conditions desired of the solution of equation (3) the following:

$$\left.\begin{aligned} & g(s, \infty) = 1 \\ & g(0, \eta) = g_0(\eta) = 1 - (1 - f_0')\,\tilde{q}(0)/f_{0,w}'' \\ & \frac{\partial g}{\partial \eta}(s, 0) = \tilde{q}(s), \text{ given but arbitrary.} \end{aligned}\right\} \tag{5}$$

The initial condition arises by considering $s\partial g/\partial s$ in equation (3) to approach zero as $s \to 0$ and by noting that a Crocco-type relation must then prevail as $s \to 0$. If the wall enthalpy expressed in terms of $g_w(0)$ is eliminated by using the definition of $\tilde{q}$ at $s = 0$, the above result is obtained.

To construct the solution of equation (3) subject to the conditions of equations (5), we solve first a related problem which may be stated as follows: Consider $\tilde{g}(s, \eta; \xi)$ where ξ is a parameter and where

$$\left.\begin{aligned} & L\tilde{g} = 0 \\ & \tilde{g}(0, \eta; \xi) = \tilde{g}(s, \infty; \xi) = 0 \\ & \frac{\partial \tilde{g}}{\partial \eta}(s, 0; \xi) = 0, \quad 0 \leqslant s < \xi \\ & \qquad = 1, \quad \xi < s. \end{aligned}\right\} \tag{6}$$

The solution is readily obtained by consideration of reference 3; we obtain

$$\left.\begin{aligned} \tilde{g}(s, \eta; \xi) &\equiv 0, \quad 0 \leqslant s < \xi \\ &= -\frac{(1 - f_0')}{f_{0,w}''} + \sum_{j=1}^{\infty} A_j \left(\frac{s}{\xi}\right)^{-\kappa_j/2} P_j(\eta), \\ &\qquad \xi < s. \end{aligned}\right\} \tag{7}$$

where the functions $P_j(\eta)$ are the eigenfunctions with related eigenvalues κ_j defined by

$$P_j'' + f_0 P_j' + \kappa_j f_0' P_j = 0$$

$$P_j'(0) = P_j(\infty) = 0 \tag{8}$$

and where the A_j coefficients are arbitrary and are determined below. That equation (7) is a solution may be confirmed by substitution into equations (6). We have changed for clarity and convenience the notation from that of reference 3 but the properties of the P_j functions have been examined there; it is shown that provided the eigenvalues are selected so that P_j approaches zero exponentially as $\eta \to \infty$, the eigenvalues are real and

* Cf. references 3 and 4 for treatments wherein the assumptions of simplified transports may be relaxed in a perturbation sense.

positive and the eigenfunctions form a complete orthogonal set with respect to functions which decay to zero exponentially as $\eta \to \infty$; the orthogonality condition is

$$\int_0^\infty \frac{f_0'}{f_0''} P_m P_n \, d\eta = \delta_{mn} C_n. \tag{9}$$

In reference 3 the first 10 eigenfunctions, eigenvalues and squares of the norms, C_n, have been given with the convenient scaling condition $P_j(0) = 1$. For completeness we present the available eigenvalues and values of C_n in Table 1.

It is convenient to employ the orthogonality property of the eigenfunctions to determine the A_j coefficients; we impose the condition that $\tilde{g}(s, \eta; \xi)$ is continuous at $s = \xi$, i.e. that

$$0 = -(1 - f_0') f_{0,w}''^{-1} + \sum_{j=1}^{\infty} A_j P_j \tag{10}$$

and obtain

$$A_j = \frac{1}{C_j} \int_0^\infty \frac{f_0'}{f_0''} \left(\frac{1 - f_0'}{f_{0,w}''}\right) P_j \, d\eta \tag{11}$$

$$= 1/C_j \kappa_j f_{0,w}''. \tag{12}$$

We thus satisfy equation (10) in an integral sense over the semi-infinite range of η but cannot expect it to be identically satisfied. The step from equation (11) to equation (12) is accomplished by substituting within the integrand for $f_0' P_j$ from equation (8) and by integrating by parts. In Table 1 the values of A_j for the first 10 values of j are given.

Table 1. Eigenvalues and related coefficients

j	κ_j	C_j	A_j
1	1	2·267	0·939
2	2·77	3·215	0·239
3	4·62	3·830	0·120
4	6·51	4·237	0·0772
5	8·41	4·609	0·0549
6	10·32	4·934	0·0418
7	12·24	5·199	0·0335
8	14·17	5·403	0·0278
9	16·10	5·600	0·0236
10	18·04	5·709	0·0207

With $\tilde{g}$ available, the solution for $\tilde{g}(s, \eta)$ may be written down in terms of a Duhamel integral, i.e.

$$g(s, \eta) = 1 - \left(\frac{1 - f_0'}{f_{,0w}''}\right) \tilde{q}(0) + \int_0^s \tilde{g}(s, \eta; \xi) \frac{d\tilde{q}}{d\xi} \, d\xi. \tag{13}$$

This may be arranged in a more convenient form; we substitute the solution for g and integrate by parts to obtain

$$g(s, \eta) = 1 + \tilde{q}(s) \left[-(1 - f_0') f_{0,w}''^{-1} + \sum_{j=1}^{\infty} A_j P_j(\eta)\right] - \sum_{j=1}^{\infty} A_j (\kappa_j/2) s^{-\kappa_j/2} P_j(\eta) \int_0^s \xi^{(\kappa_j/2)-1} \tilde{q}(\xi) \, d\xi. \tag{14}$$

The distribution of wall enthalpy corresponding to the prescribed distribution of $\tilde{q}$ is obtained from equation (14) by setting $\eta = 0$; thus

$$g_w(s) = 1 + \tilde{q}(s) \left[- f_{0,w}''^{-1} + \sum_{j=1}^{\infty} A_j\right] - \sum_{j=1}^{\infty} A_j (\kappa_j/2) s^{-\kappa_j/2} \int_0^s {}^{(\kappa_j \xi/2)-1} \tilde{q}(\xi) \, d\xi \tag{15}$$

Equations (14) and (15) are the desired solution, "in principle" exact, but restricted "in practice" because of the truncation of the summation. The solution for $g_w(s)$ is formally quite different from that given in reference 1, but of course the two may be in good numerical agreement for a variety of cases.

CONCLUDING REMARKS

It is interesting to consider several special cases treated in reference 1. We assume $\tilde{q} = \tilde{q}_0 \, s^{n+1/2}$ so that $q_w \sim s^n$ where n may or may not be an integer. Then equation (15) yields readily

$$g_w = 1 + \tilde{q}_0 \, s^{n+1/2} \left[-\frac{1}{f_{0,w}''} + \sum_{j=1}^{\infty} A_j \left(\frac{\kappa_j}{\kappa_j + 1 + 2n}\right)\right]. \tag{16}$$

Qualitatively this is in accord with reference 1; thus $q_w \sim s^n$ corresponds to $g_w - 1 \sim s^{n+1/2}$.

For the step function in energy transfer, i.e. $\tilde{q} = 0$ for $0 \leqslant s < s_0$, $\tilde{q} = \tilde{q}_0 =$ constant for $s > s_0$, we obtain from equation (15)

$$g_w(s) = 1 \quad ,0 \leqslant s < s_0$$

$$= 1 + \tilde{q}_0 \left[-f_{0,w}''^{-1} + \sum_{j=1}^{\infty} A_j (s_0/s)^{\kappa_j/2}\right], \; s > s_0. \tag{17}$$

Again this solution is formally quite different from the approximate solution in reference 1, although numerically they might be in good agreement.

It is beyond the desired scope of this note to carry out in detail the comparisons of the approximate analysis with the present one. It does appear, however, that the above solution and its related solution given in reference 3, namely the energy distribution, including the heat transfer, for arbitrary $g_w(s)$, would be useful for solving problems involving simultaneous laminar convection and radiative heat transfer. We note that for problems involving chemical reaction, surface catalysis, and/or heterogeneous composition, the solution for $g_w(s)$ for prescribed energy flux does not yield directly the surface temperature distribution but rather must be combined with solutions of the equations of species conservation to yield this distribution. Some interesting problems for further study would appear to be involved in these cases.

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